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Classical parastochastics and quantum mechanics: some observations

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Abstract. Modified commutation relations which have been used in a classical stochastic context are shown to be similar, in a simple quantum mechanical example, to the modification of the potential. In quantum field theory their effect is to yield a nonlinear field equation for a free scalar boson. It is concluded that such modified commutation relations are incompatible with quantum mechanics.

In a recent article (Frisch and Bourret 1970) modified commutation relations of the form

$$\boldsymbol{a}\boldsymbol{a}^{\dagger} - \lambda \boldsymbol{a}^{\dagger}\boldsymbol{a} = 1 \qquad |\lambda| \leqslant 1 \tag{1}$$

were introduced to provide the 'parastochastic' operator† representation of the stochastic variable x according to

$$\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{a}^{\dagger}. \tag{2}$$

The usual relations

$$\langle 0|\boldsymbol{a}^{\dagger} = \boldsymbol{a}|0\rangle = 0 \qquad \langle 0|0\rangle = 1 \tag{3}$$

are retained, and the term parastochastic reflects the possibility of computing 'vacuum' expectation values of polynomial functions of x

$$F(\boldsymbol{x}) = \sum_{k} c_k \boldsymbol{x}^k \tag{4}$$

which coincide with a classical evaluation using a suitable lambda-dependent probability distribution function $P(x:\lambda)$, that is

$$\langle 0|F(\boldsymbol{x})|0\rangle = \int F(x)P(x:\lambda)\,\mathrm{d}x. \tag{5}$$

For the special cases of $\lambda = 1, 0, -1$ the function $P(x:\lambda)$ becomes Gaussian, semicircular and dichotomic, respectively. For intermediate values a numerical determination of $P(x:\lambda)$ has been made and is graphically presented in the article mentioned. For the three values of lambda cited useful applications of parastochastics operators have been found in the study of stochastically perturbed classical systems.

† Parastochastics is not to be identified with *parastatistics* (Greenberg and Messiah 1965), which involve trilinear commutation relations compatible with the Heisenberg equations of motion.

The question naturally arises: could these modified commutation relations be introduced into quantum mechanics meaningfully, and if so what would be the consequences?

To the first question the answer must be no since it is readily shown that the classical equations of motion cannot be obtained from Heisenberg's equations. This means that the Hamiltonian, taken from the classical problem, does not describe the evolution of the state vector and the Heisenberg representation of the dynamical variables is no longer valid. If we accept, however, that the equations of motion derived from the classical Hamiltonian need not coincide with the classical equations of motion, then one can salvage a kind of quantum mechanics none the less. We would like to describe some of its more curious features.

Consider the harmonic oscillator, described in terms of the variables

$$\boldsymbol{a} = (2m\hbar\omega)^{-1/2}(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{x}) \qquad a^\dagger = a^*. \tag{6}$$

It's Hamiltonian is

$$\boldsymbol{H} = \tfrac{1}{2}\hbar\omega(\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}) \tag{7}$$

which, in view of (1), may be written

$$\boldsymbol{H} = \tfrac{1}{2}\hbar\omega\{1 + (1+\lambda)\boldsymbol{a}^\dagger\boldsymbol{a}\}. \tag{8}$$

Using the representation for the operators $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ given in the paper cited it is shown without difficulty that the eigenvalues of the Hamiltonian operator are given by

$$E_n(\lambda) = \tfrac{1}{2}\hbar\omega\left(1 + \frac{(1+\lambda)(1-\lambda^n)}{1-\lambda}\right) \qquad n = 1, 2, \ldots \tag{9}$$

$$E_0(\lambda) = \tfrac{1}{2}\hbar\omega \qquad n = 0.$$

The appearance of these levels for a value of lambda between zero and one is shown semiquantitatively in figure 1. It is somewhat as if the potential well had been modified so as to resemble the Morse potential. An assembly of such systems exhibits a 'Schottky anomaly' as the thermal energy approaches the maximum energy permitted by the discrete energy formula given above. This is readily shown by a numerical evaluation of the classical partition function $Z = \Sigma \exp(-\beta E_n(\lambda))$. The calculation has been made for the value $\lambda = \frac{1}{2}$ and the resulting specific heat curve is shown in figure 2.

Despite the break with the usual structure of quantum mechanics the modified harmonic oscillator described above appears to be sufficiently 'physical' to suggest that

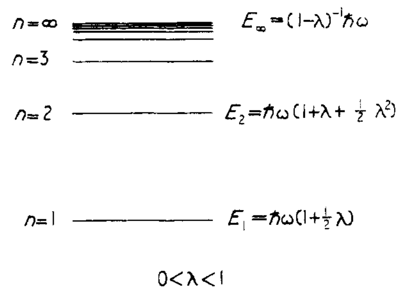

Figure 1. Boson energy levels (schematic).

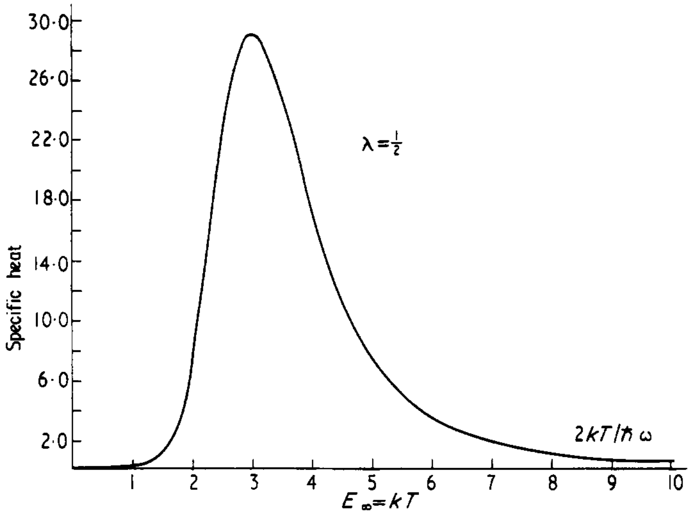

Figure 2. Boson specific heat curve.

some effects of potentials might be usefully simulated by an appropriate choice of lambda.

The field theoretic equivalent of what we have just described consists of considering, for example, the neutral boson field satisfying (in 1 dimension)

$$\Box\phi = \mu^2\phi \qquad \omega_k^2 = k^2+\mu^2 \tag{10}$$

and applying the modified commutation relations to the operators a_k and $a_k^\dagger$ appearing in the expansion

$$\phi = \phi^{(+)}+\phi^{(-)} = \sum_k (a_k f_k + a_k^\dagger f_k^*) \tag{11}$$

in which the f_k are given by

$$f_k(x,t) = (2\omega_k L)^{-1/2}\exp(\mathrm{i}kx-\mathrm{i}\omega_k t) \qquad \hbar = c = 1. \tag{12}$$

The operators a_k and $a_k^\dagger$ may be recovered from the wavefunction (Lurié 1968), if we define the 'bilateral' time derivative $\overleftrightarrow{\partial}$ by

$$A\overleftrightarrow{\partial}B \equiv A\frac{\partial B}{\partial t}-\frac{\partial A}{\partial t}B \tag{13}$$

and the special inner product given by

$$((A,B)) \equiv \mathrm{i}\int A^*\overleftrightarrow{\partial}B\,\mathrm{d}x. \tag{14}$$

We then have

$$a_k = ((f_k,\phi)) \quad \text{and} \quad a_k^\dagger = ((\phi,f_k)). \tag{15}$$

The Hamiltonian for the system is

$$H = \tfrac{1}{2}\sum_{k}(a_k^\dagger a_k + a_k a_k^\dagger)\omega_k \tag{16}$$

which reduces, for general λ, to

$$H = \tfrac{1}{2}(1+\lambda)\sum_{k}\omega_k a_k^\dagger a_k \tag{17}$$

dropping the scalar part corresponding to vacuum fluctuations. The modified commutation relations now take the slightly more general form

$$a_k a_j^\dagger - \lambda a_j^\dagger a_k = \delta_{kj}. \tag{18}$$

Using these we attempt to reconstruct the wave equation by means of the Heisenberg equation of motion

$$\mathrm{i}\frac{\partial \phi^{(\pm)}}{\partial t} = (\phi^{(\pm)}, H). \tag{19}$$

Using the notation

$$\Omega \equiv \left(-\frac{\partial^2}{\partial x^2}+\mu^2\right)^{1/2} \tag{20}$$

we may write the results as follows:

$$\mathrm{i}\frac{\partial \phi^{(+)}}{\partial t} = \tfrac{1}{2}(1+\lambda)\Omega\phi^{(+)} - \tfrac{1}{2}(1-\lambda^2)\phi^{(+)}\sum_{\beta}\omega_\beta|((\phi, f_\beta))|^2 \tag{21}$$

$$\mathrm{i}\frac{\partial \phi^{(-)}}{\partial t} = -\tfrac{1}{2}(1+\lambda)\Omega\phi^{(-)} + \tfrac{1}{2}(1-\lambda^2)\phi^{(-)}\sum_{\beta}\omega_\beta|((\phi, f_\beta))|^2. \tag{22}$$

The first terms on the right, despite their unusual appearance, differ only by the $\frac{1}{2}(1+\lambda)$ factor from the conventional ones. The second terms, being trilinear and creating a coupling between the $\phi^{(+)}$ and $\phi^{(-)}$ components of the field are of course entirely foreign to the behaviour of a free field as normally understood.

References

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